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ALLEE EFFECT ON A NEW DELAY POPULATION MODEL AND STABILITY ANALYSIS

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Abstract

In this study, we handle a new delay population model and investigate stability of this model with and without Allee effect. In both cases, stability conditions are given.

1. Introduction

In late years, Allee effect has been a very important research topic in many areas of ecology and biology. The Allee effect that is first introduced by Allee [1] in 1931, refer to inverse density dependence at low density. Allee effect is biological phenomenon characterized by a positive correlation between population density and the per capita population growth rate in very small population. This effect may impact the form of reducing or increasing the population, such as inbreeding depression, food exploitation, social dysfunction at small population size, predator avoidance of defence and difficulties finding in mates, and individual

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fitness. Authors have studied the stability of different population models within the framework of effect like this [3, 5, 6].

Previous studies indicate that Allee effect has different effects on different populations. Mathematical formulations of the population and the factors effect population will provides with information in the future about the development of that group of living beings.

In this present study, our purpose is to investigate the stability of equilibrium point with and without Allee effect for the case when $T = 1$ and compare the stability of these models. Let us look at the general nonlinear delay difference equation representing the dynamics of a population

$$N_{t+1} = \lambda N_t f(N_t, N_{t-T}), \quad \lambda > 0, \tag{1}$$

where λ is per capita growth rate, which is always positive; N_t represents the population density at time t; T is the time for sexual maturity; and $f(N_t, N_{t-T})$ is the function describing interactions (competitions) among mature individuals. It is generally assumed that f continuously decreases as density increases.

We assume that f satisfies the following conditions:

(1) $\frac{\partial f}{\partial N_t}(N, N) < 0, \quad \frac{\partial f}{\partial N_{t-1}}(N, N)$ for $N \in [0, \infty)$;

(2) $f(0, 0)$ is a positive finite number.

Assume that Equation (1) has an equilibrium points as N^*.

2. Main Result

In this section, we studied on delay difference model under a competitive effect. Firstly, we obtained the stability conditions of the positive equilibrium point of Equation (1) with and without Allee effect. We compared the stability of these models. In conclusion, we observed that Allee effect increases stability in the model class, we studied and confirm result with numerical simulation.

2.1. Stability analysis on the discrete delay model (1) without Allee effects

Theorem 1. *The equilibrium point* N^* *of Equation* (1) *is locally stable*, *if and only if the inequalities*

$$N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} > -1, \tag{2}$$

$$N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)} < 2, \tag{3}$$

hold.

Proof. By the equilibrium point definition, we write

$$1 = \lambda f(N^*, N^*). \tag{4}$$

Let us take $p = F_{N_t}(\lambda, N^*, N^*)$, $q = F_{N_{t-1}}(\lambda, N^*, N^*)$. Then, if we consider the equality (4) and Equation (1) that

$$p = 1 + N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)}, \quad q = N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}.$$

We conclude that N^* is locally stable, if and only if

$$|p| < 1 - q < 2, \tag{5}$$

with the help of Schur-Cohn criteria [2, 4]. From where, we have

$$\left|1 + N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)}\right| < 1 - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 2.$$

If we write the last expression in two parts, we obtain

$$\left|1 + N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)}\right| < 1 - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)},$$

$$N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} > -1.$$

From the first inequality, we get

$$\left[N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)} \right] < 2,$$

$$\left[N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} + N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)} \right] < 0.$$

The last inequality is always provide for $f_{N_t} < 0$, $f_{N_{t-1}} < 0$. So the proof is completed. ■

2.2. Stability analysis on the discrete delay model (1) with Allee effect

We study the stability analysis of the equilibrium points of Equation (1) with the addition of Allee effect at time t and $t-1$.

2.2.1. Allee effect at time *t*

We consider the following nonlinear delay difference equation by the addition of Allee effect to discrete delay model (1)

$$N_{t+1} = F^{(\alpha^-)}(\lambda^*, N_t, N_{t-1}) = \lambda^* N_t \alpha(N_t) f(N_t, N_{t-1}), \quad \lambda^* > 0, \qquad (6)$$

where the function f satisfies the conditions (1) and (2). The conclusion of the biological facts requires the following assumption on α:

(3) If $N = 0$, then $\alpha(N) = 0$; that is, there is no reproduction without partners.

(4) $\alpha'(N) > 0$ for $N \in (0, \infty)$; that is, Allee effect decreases as density increases.

(5) $\lim_{N \to \infty} \alpha(N) = 1$; that is, Allee effect vanishes at high densities.

Equation (6) has at most two positive equilibrium points so that it has the conditions (1)-(5).

Then we get the following theorem:

Theorem 2. *The equilibrium point* N^* *of Equation* (6) *is locally stable*, *if and only if the inequality*

$$N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} > -1, \tag{7}$$

$$N^* \left[\frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - \frac{\alpha'(N^*)}{\alpha(N^*)} - \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)} \right] < 2, \tag{8}$$

hold.

Proof. Likewise, if p and q values are calculated for Equation (6), we obtain

$$p = F_{N_t}(\lambda, N^*, N^*) = 1 + N^* \frac{\alpha'(N^*)}{\alpha(N^*)} + N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)};$$

$$q = F_{N_{t-1}}(\lambda, N^*, N^*) = N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)}.$$

For the inequality in (5), we obtain

$$\left| 1 + N^* \frac{\alpha'(N^*)}{\alpha(N^*)} - N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)} \right| < 1 - N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 2,$$

$$N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} > -1;$$

and

$$N^* \left[\frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - \frac{\alpha'(N^*)}{\alpha(N^*)} - \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)} \right] < 2;$$

$$N^*\left[\frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} + \frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)}\right] < 0.$$

The following expression always hold true from definition of f and α. Consequently, we confirm (7) and (8). ■

Theorem 3. *Allee effect at time t increases the stability of Equation* (1).

Proof. Let us take $x := N^* \dfrac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 0$, $y := N^* \dfrac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)} < 0$, and $z := N^* \dfrac{\alpha'(N^*)}{\alpha(N^*)} > 0$. Equation (1) is stable, if and only if

$$x > -1 \text{ and } x - y < 2. \tag{9}$$

If x, y, and z expressions are written in the Equations (7) and (8), we obtain stability conditions (10), respectively,

$$x > -1 \text{ and } x - y - z < 2. \tag{10}$$

The last inequality is provide in (10) for $x - y < 2\,(\forall z > 0)$. It is clear that the stability of Equation (1) increases. ■

2.2.2. Allee effect at time t –1

We now incorporate an Allee effect into the discrete delay model (1) as follows:

$$N_{t+1} = F^{(\alpha^+)}(\lambda^*, N_t, N_{t-1}) = \lambda^* N_t \alpha(N_{t-1}) f(N_t, N_{t-1}). \tag{11}$$

Then we can state the following theorem:

Theorem 4. *The equilibrium point* N^* *of Equation* (11) *is locally stable, if and only if the inequality*

$$N^* \frac{\alpha'(N^*)}{\alpha(N^*)} + N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} > -1, \tag{12}$$

$$N^* \frac{\alpha'(N^*)}{\alpha(N^*)} + N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)} < 2, \tag{13}$$

hold.

Proof. According to Equation (11), p and q expressions are as follows:

$$p = F_{N_t}(\lambda, N^*, N^*) = 1 + N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)};$$

$$q = F_{N_{t-1}}(\lambda, N^*, N^*) = N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} \right].$$

For N^* equilibrium point to be stable, if the inequalities in (5) are evaluated according to p and q values, then we get

$$\left| 1 + N^* \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)} \right| < 1 - N^* \frac{\alpha'(N^*)}{\alpha(N^*)} + N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 2;$$

$$N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} - \frac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)} \right] < 2;$$

$$N^* \left[\frac{\alpha'(N^*)}{\alpha(N^*)} + \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} + \frac{f_{N_{t-2}}(N^*, N^*)}{f(N^*, N^*)} \right] < 0;$$

and

$$N^* \frac{\alpha'(N^*)}{\alpha(N^*)} + N^* \frac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} > -1.$$

So the proof is completed. ■

Theorem 5. *Allee effect at time* $t - 1$ *increases the stability of Equation* (1).

Proof. Let us take $x := N^* \dfrac{f_{N_{t-1}}(N^*, N^*)}{f(N^*, N^*)} < 0,\ y := N^* \dfrac{f_{N_t}(N^*, N^*)}{f(N^*, N^*)}$ < 0, and $z := N^* \dfrac{\alpha'(N^*)}{\alpha(N^*)} > 0$. Equation (1) is stable, if and only if conditions (9) provide that

$$x > -1 \text{ and } y > -3. \tag{14}$$

If x, y, and z expressions are written in the Equations (12) and (13), we obtain stability conditions (15), respectively,

$$x + z > -1 \text{ and } x - y + z < 2. \tag{15}$$

If the last two inequality in (15) are solved, we have

$$x + z > -1 \text{ and } y > -3. \tag{16}$$

From first inequality in (14) and (16), then we can say that

$$x < x + z.$$

Consequently, it is clear that the stability of Equation (1) increases. ■

Corollary 6. *Allee effect at time $t-1$ has more impact stability than Allee effect at time t.*

2.3. Numerical simulations

In Figures 1 and 2, we graph the 2D trajectories of the population dynamics models (1), respectively, with and without Allee effect at time $t, t-1$. Finally, Figure 3 is formed by combining the Figures 1 and 2. This figures we take the function $f(N_t, N_{t-1}) = (1 - N_t / K - N_{t-1} / K)$ and the Allee function $\alpha(N_i) = N_i / (\alpha + N_i)$, $i = t-1, t$, where α is a positive constant. It is obvious from the graph that the comparisons of the population density diagrams also verify the stabilizing impact of the Allee effects.

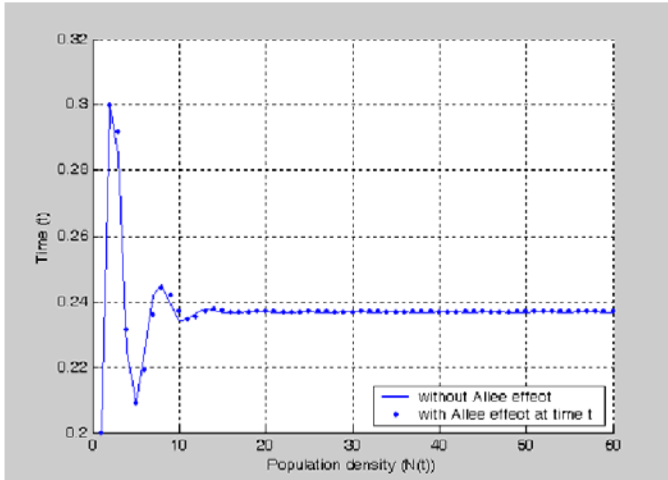

Figure 1. Density-time graphs model. $N_{t+1} = \lambda N_t(1 - N_t / K - N_{t-1} / K)$ and $N_{t+1} = \lambda^* \alpha(N_t) N_t(1 - N_t / K - N_{t-1} / K)$ with $\lambda = 1.9$, $K = 1$, $\alpha(N_t) = N_t / (\alpha + N_t)$, $\alpha = 0.03$, $\lambda = \lambda^* \alpha(N^*)$ and the initial conditions $N_{-1} = 0.2$, $N_0 = 0.3$.

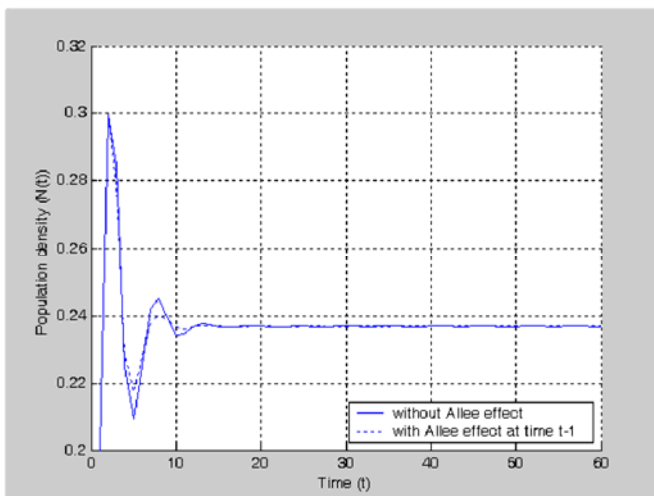

Figure 2. Density-time graphs model. $N_{t+1} = \lambda N_t(1 - N_t / K - N_{t-1} / K)$ and $N_{t+1} = \lambda^* \alpha(N_t) N_t(1 - N_t / K - N_{t-1} / K)$ with $\lambda = 1.9$, $K = 1$, $\alpha(N_{t-1}) = N_{t-1} / (\alpha + N_{t-1})$, $\alpha = 0.03$, $\lambda = \lambda^* \alpha(N^*)$ and the initial conditions $N_{-1} = 0.2$, $N_0 = 0.3$.

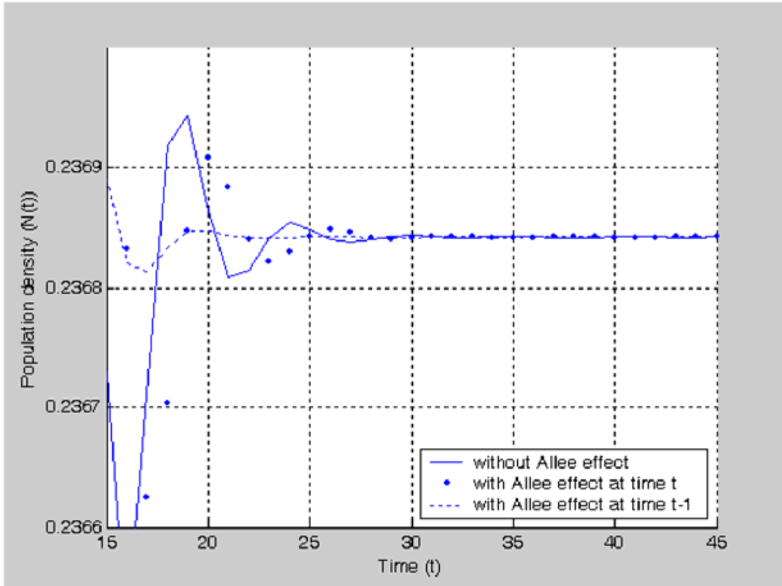

Figure 3. Density-time graphs model. $N_{t+1} = \lambda N_t(1 - N_t / K - N_{t-1} / K)$ and $N_{t+1} = \lambda^* \alpha(N_t) N_t(1 - N_t / K - N_{t-1} / K)$ with $\lambda = 1.9$, $K = 1$, $\alpha(N_{t-1}) = N_{t-1} / (\alpha + N_{t-1})$, for $t, t-1 : \alpha = 0.03$, $\lambda = \lambda^* \alpha(N^*)$ and the initial conditions $N_{-1} = 0.2$, $N_0 = 0.3$.

3. Conclusion and Discussion

The balance in live life is essential for continuity of life. Accordingly, it is significant to have information about that population. This situation attracts attention on dynamic population models. We know there is more than one variable having effect on population. What are the effects of these variables on population? “Or can demanded population be built up when required?” questions spring to mind. This may not be easy. Because the factors undertaken may affect the population in a negative or positive way. There, survival situations of kinds are the results of Allee effect, which they were exposed.

In this study, we saw that the population model has a stable structure. Addition of Allee effect expedited the stability. This situation is a result of usability of the population.

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